RIGOROUS COMPUTATIONS OF HOMOCLINIC TANGENCIES *

ZIN ARAI† AND KONSTANTIN MISCHAIKOW‡

Abstract. In this paper, we propose a rigorous computational method for detecting homoclinic tangencies and structurally unstable connecting orbits. It is a combination of several tools and algorithms, including the interval arithmetic, the subdivision algorithm, the Conley index theory, and the computational homology theory. As an example we prove the existence of generic homoclinic tangencies in the Hénon family.

Key words. homoclinic tangency, connecting orbit, Conley index, computational homology

AMS subject classifications. 37B30, 37G25, 37M20

1. Introduction.

In this paper, we present a method for proving the existence of homoclinic tangencies and structurally unstable connecting orbits. More precisely we are interested in proving the existence of *generic* tangencies in a one-parameter family of maps; that is, a quadratic tangency that unfolds generically in the family. The importance of the generic homoclinic tangency comes from the fact that it implies the occurrence of the Newhouse phenomena [17] and strange attractors [12].

To explain how the method works, we apply it to the Hénon family

$$\begin{aligned} H_{a,b} &: \mathbb{R}^2 \to \mathbb{R}^2 \\ (x, y) &\mapsto (a - x^2 + by, x) \end{aligned} \tag{1.1}$$

Belief in the existence of homoclinic tangencies in the Hénon family is easily obtained by numerical experiments. For example, the plots in Figure 1.1, suggest the existence of tangencies for parameter values close to $a = 1.4$, $b = 0.3$ and $a = 1.3$, $b = -0.3$. Our motivation for this work is to develop a general computationally inexpensive method that provides a mathematically rigorous verification of this numerically induced speculation. In fact, using this technique we prove the following two results.

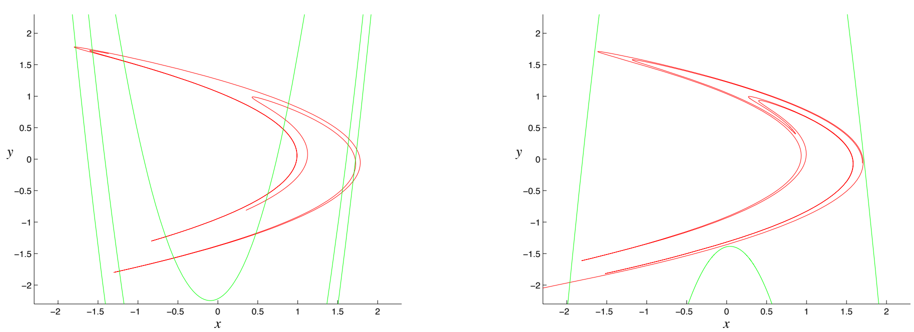

FIG. 1.1. *Left*: $a = 1.4, b = 0.3$; *Right*: $a = 1.3, b = -0.3$

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Theorem 1.1. *Fix any* b_0 *sufficiently close to* 0.3*. Then there exists*

$$a \in [1.392419807915, 1.392419807931]$$

such that the one-parameter family H_{a,b_0} *has a generic homoclinic tangency with respect to the saddle fixed point on the first quadrant.*

Theorem 1.2. *Fix any* b_0 *sufficiently close to* -0.3*. Then there exists*

$$a \in [1.314527109319, 1.314527109334]$$

such that the one-parameter family H_{a,b_0} *has a generic homoclinic tangency with respect to the saddle fixed point on the third quadrant.*

Here we say that a tangency in a one-parameter family is generic if the intersection of unstable and stable manifolds are quadratic, and the intersection is unfolded generically in the family (see Section 2 for the precise definition).

Similar results can also be attained by a complex analytic method of Fornæss and Gavosto [4, 5]. Compared to their method, which depends on the analyticity of maps, our method is rather geometric and topological, and is designed so that it can be applied to a wider class of maps. Essentially, we require a continuous family of C^2 diffeomorphisms for which we can compute the image of the maps using interval arithmetic. We present a brief overview of our approach; a more detailed description is provided in the following sections.

The essential difficulty of a computer assisted proof in dynamics is that the dynamical system the computer is capable of representing and evaluating is at best a small perturbation of the system of interest. However, the small perturbations can induce bifurcations which create or destroy the dynamical structure of interest. The Conley index [6, 10, 11] is a powerful tool for this type of problem precisely because it remains constant under perturbations. It is an algebraic topological quantity which can be used to prove the existence of particular dynamical structures including connecting orbits. Using recently developed computational topology tools [8] it is possible to compute the index from the numerically generated data with the guarantee that the index is valid for the original system of interest.

To be more precise in our discussion consider $f : X \to X$ a continuous map on a locally compact metric space X. We use the homological Conley index with integer coefficients defined for an isolated invariant set S of f, and denote it by $\mathrm{Con}_*(S, f)$ or simply by $\mathrm{Con}_*(S)$. Recall that $\mathrm{Con}_*(S)$ is the shift equivalence class of the pair of a graded module $CH_*(S)$ and an endomorphism $\chi_*(S)$ on $CH_*(S)$. (See [6, 8] for the concept of shift equivalence and the definition of the Conley index for maps.) By an abuse of notation we write the shift equivalent class $[(CH_*(S), \chi_*(S))]$ simply as $(CH_*(S), \chi_*(S))$.

We say an orbit $\sigma : \mathbb{Z} \to X$, $f(\sigma(k)) = \sigma(k + 1)$ for all k, is a connecting orbit from S_1 to S_2 if its α-limit set is contained in S_1 and its ω-limit set is contained in S_2. The maximal invariant set of $N \subset X$ will be denoted by $\mathrm{Inv}(N)$.

The following theorem which is proven in Section 3 lies at the heart of our algebraic machinery to find connecting orbits.

Theorem 1.3. *Let* N_1, N_2 *and* N *be isolating neighborhoods and assume* N *is the disjoint union of* N_1 *and* N_2*. If* $f(N_2) \cap N_1 = \emptyset$ *and*

$$\mathrm{Con}_*(\mathrm{Inv}(N)) \ncong \mathrm{Con}_*(\mathrm{Inv}(N_1)) \oplus \mathrm{Con}_*(\mathrm{Inv}(N_2))$$

as shift equivalence classes, then there exists a connecting orbit from $\mathrm{Inv}(N_1)$ *to* $\mathrm{Inv}(N_2)$*.*

Consider a continuous one-parameter family of C^2 diffeomorphisms $f_\lambda : X \to X$ where the parameter $\lambda \in \mathbb{R}$ and assume that f_0 has a homoclinic tangency. Generically one expects that for $\lambda \neq 0$, f_λ will not posses a homoclinic tangency. Since the Conley index is robust with respect to perturbations there is no hope that an existence proof can be obtained by a direct application of the index. Thus, we need to recast the problem in such a way that generic homoclinic tangencies become robust isolated objects.

To obtain the isolation observe that at a homoclinic tangency the stable and unstable manifolds share a tangent vector. Let $Pf_\lambda : PX \to PX$ be the induced map on the projective bundle of X. Then a homoclinic tangency of f_0 corresponds to a connecting orbit of Pf_0.

The robustness can be obtained by considering the entire family of maps simultaneously. To do this define

$$\begin{aligned} F : X \times \mathbb{R} &\to X \times \mathbb{R} \\ (x, \lambda) &\mapsto (f_\lambda(x), \lambda) \end{aligned} \tag{1.2}$$

One now expects that if $\bar{F} : X \times \mathbb{R} \to X \times \mathbb{R}$, of the form $\bar{F}(x, \lambda) = (\bar{f}_\lambda(x), \lambda)$, is induced by a perturbation of f, then there exists $\lambda_0 \approx 0$ such that $\bar{f}_{\lambda_0}$ posses a homoclinic orbit.

With this in mind one is tempted to apply Theorem 1.3 by restricting F to Λ a compact interval containing 0 and computing the index of PF. Unfortunately, for technical reasons explained in Section 3 this does not work. Instead we compute using PF' where $F'X \times \mathbb{R} \to X \times \mathbb{R}$ represents a perturbation of F with the property that $F = F' \mid_{X \times \Lambda}$. Thus a heteroclinic tangency for F' is equivalent to a heteroclinic tangency for F and hence f_λ for some $\lambda \in \Lambda$.

To check that the heteroclinic tangency is indeed quadratic it is sufficient to show that the heteroclinic orbit does not define a connecting orbit for $PPf : PPX \to PPX$ the induced map on the projective bundle of PX.

The details concerning the induced dynamics on the projective bundles is described in Section 2. The Conley index tools are described in Section 3. Finally in Section 4 we indicate how these techniques are implemented in the context of the Hénon family. All the source files used in the computation can be downloaded from `http://www.math.kyoto-u.ac.jp/~arai`. To run the computation, one needs software packages GAIO [2, 3] and Computational Homology Programs (`CHomP`, [14]).

2. Tangencies and connecting orbits. Let f be a diffeomorphism on a manifold X. We denote the tangent bundle of X by TX and the differential of f by Tf, as usual.

From the dynamical system $f : X \to X$, we derive a new dynamical system $Pf : PX \to PX$ which is defined as follows. The space PX is the projective bundle associated to the tangent bundle of X, that is, the fiber bundle on X whose fiber over $x \in X$ is the projective space of $T_x X$. Namely,

$$PX = \coprod_{x \in X} P_x X := \coprod_{x \in X} \{\text{one-dimensional subspace of } T_x X\}.$$

Define Pf to be the map induced from Tf on PX, namely, $Pf([v]) := [Tf(v)]$ where $0 \neq v \in TX$ and $[v]$ is the subspace spanned by v. Identifying X with the image of the zero section of TX, we have the following commutative diagram:

$$\begin{array}{ccc} TX \setminus X & \xrightarrow{Tf} & TX \setminus X \\ \pi \downarrow & & \downarrow \pi \\ PX & \xrightarrow{Pf} & PX \\ \pi' \downarrow & & \downarrow \pi' \\ X & \xrightarrow{f} & X. \end{array}$$

Let $p \in X$ be a hyperbolic fixed point of f and $T_pX = \tilde{E}^s_p \oplus \tilde{E}^u_p$ the corresponding splitting of the tangent space. We denote the stable and unstable manifolds of p by $W^s(p)$ and $W^u(p)$, respectively.

Define $E^s_p := \pi(\tilde{E}^s_p \setminus \{0\})$ and $E^u_p := \pi(\tilde{E}^u_p \setminus \{0\})$. The spaces E^s_p and E^u_p are isolated invariant sets with respect to $Pf : PX \to PX$.

THEOREM 2.1 (Proposition 5.3 of [1]). *Let p, q be hyperbolic fixed points of f, and assume that $\dim W^u_f(p) + \dim W^s_f(q) \leq n$. If there exists a connecting orbit from E^u_p to E^s_q under Pf, then $W^u_f(p)$ and $W^s_f(q)$ have a non-transverse intersection.*

Note that if $p = q$, the case of a homoclinic orbit, $\dim W^u_f(p) + \dim W^s_f(p) = n$ always holds. Therefore, our problem of finding homoclinic tangencies is now translated to that of finding connecting orbits from $E^u(p)$ to $E^s(p)$ with respect to $Pf : PX \to PX$.

Next, we discuss genericity of tangencies. The definition of genericity is taken from [9]. It is a generalization of the generic (or non-degenerate) tangencies for surface diffeomorphism (see [4, 5, 12, 17]).

Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a one-parameter family of C^2 diffeomorphism depending smoothly on the parameter $\lambda \in \Lambda \subset \mathbb{R}$. For simplicity, we consider the homoclinic tangency of a family of hyperbolic fixed points $p(\lambda)$ of f_λ. The case for a hyperbolic periodic point is quite similar. Assume that $p(\lambda_0)$ has a homoclinic tangency at $x \in M$ for $\lambda_0 \in \operatorname{int}\Lambda$. For simplicity, we assume that $\lambda_0 = 0$. We say that x is a 1-tangential quadratic homoclinic tangency if the tangent spaces of $W^u_{f_0}(p(0))$ and $W^s_{f_0}(p(0))$ at x has a common subspace of dimension one and the intersection of these manifolds at x is quadratic along this common 1-dimensional subspace. Now we define

$$\mathcal{W}^u_\Lambda = \bigcup_{\lambda\in\Lambda} W^u_{f_\lambda}(p(\lambda)), \quad \mathcal{W}^s_\Lambda = \bigcup_{\lambda\in\Lambda} W^s_{f_\lambda}(p(\lambda)).$$

These are smooth submanifolds of $X \times \Lambda$ (see [9]) and since we assume that x is a homoclinic tangency for f_0, they have a intersection at $(x, 0)$.

DEFINITION 2.2. *We say that $\{f_\lambda\}$ has a generic homoclinic tangency at x if x is 1-tangential quadratic tangency, and $\mathcal{W}^u_\Lambda$ and $\mathcal{W}^s_\Lambda$ are transversal in $X \times \Lambda$ at $(x, 0)$.*

We then consider the projection $(\pi', id) : PX \times \Lambda \to X \times \Lambda$ and the lifting of a generic homoclinic tangency. Let

$$PF : (x, \lambda) \mapsto (Pf_\lambda(x), \lambda) : PX \times \Lambda \to PX \times \Lambda.$$

Then it is easy to see that the sets $\mathcal{E}^u_\Lambda := \bigcup_{\lambda\in\Lambda} E^u_{p(\lambda)}$ and $\mathcal{E}^s_\Lambda := \bigcup_{\lambda\in\Lambda} E^s_{p(\lambda)}$ are normally hyperbolic invariant manifolds with respect to PF and we have

$$W^u_{PF}(\mathcal{E}^u_\Lambda) = \bigcup_{\lambda\in\Lambda} W^u_{Pf_\lambda}(E^u_{p(\lambda)}), \quad W^s_{PF}(\mathcal{E}^s_\Lambda) = \bigcup_{\lambda\in\Lambda} W^s_{Pf_\lambda}(E^s_{p(\lambda)}).$$

In this setting, the genericity of a tangency is expressed as follows.

THEOREM 2.3. *Let f_λ be a one-parameter family of diffeomorphisms with hyperbolic fixed point $p(\lambda)$ and assume f_0 has a homoclinic tangency with respect to $p(0)$. If the corresponding intersection of $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ is transversal in $PX \times \Lambda$, then the tangency is generic.*

Proof. We denote the unstable and stable dimension of $p(\lambda)$ by k and ℓ, respectively. Then $E^u_{p(\lambda)}$ and $E^s_{p(\lambda)}$ are normally hyperbolic manifold of dimension $k-1$ and $\ell - 1$. Since $E^u_{p(\lambda)}$ is contracting under $Pf_\lambda|_{P_{p(\lambda)}X}$, its unstable dimension is k and therefore $W^u_{Pf_\lambda}(E^u_{p(\lambda)})$ is $k-1+k = 2k-1$ dimensional manifold. It follows that $\dim W^u_{PF}(\mathcal{E}^u_\Lambda) = 2k$ and by the same argument, $\dim W^s_{PF}(\mathcal{E}^s_\Lambda) = 2\ell$. Since $k + \ell = \dim X$ and $\dim(PX \times \Lambda) = 2 \cdot \dim X$, this implies that the transversal intersection of $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ is 0-dimensional and hence isolated.

Now we prove the theorem.

Assume that the tangency is not 1-tangential, that is, the dimension of the intersection of the tangent spaces of $W^u_{f_0}(p(0))$ and $W^u_{f_0}(p(0))$ at x is greater than or equal to 2. It follows that the corresponding intersection of $W^u_{Pf_0}(E^u_{p(0)})$ and $W^s_{Pf_0}(E^s_{p(0)})$ must contain a copy of $\mathbb{R}P^k$ where $k \geq 1$. And therefore, the intersection of $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ can not be an isolated point. This is a contradiction.

Next, assume that the tangency is not quadratic. Then we can take smooth curves c^u on $W^u_{f_0}(p(0))$ and c^s on $W^s_{f_0}(p(0))$ through x such that they have the same first and second derivatives at x. These curves give rise to curves $\tilde{c}^u$ and $\tilde{c}^s$ on $W^u_{Pf_0}(E^u_{p(0)})$ and $W^s_{Pf_0}(E^s_{p(0)})$ that intersect at (x, θ) where θ is the direction of the tangency. Since the second derivatives of c^u and c^s are equal, $\tilde{c}^u$ and $\tilde{c}^s$ are tangent at (x, θ). This is a contradiction.

Finally, since the tangent spaces of $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ span $T_{(x,\theta,0)}(PX \times \Lambda)$ by assumption, it follows that the tangent spaces of $\mathcal{W}^u_\Lambda$ and $\mathcal{W}^s_\Lambda$ span $T_{(x,0)}(X \times \Lambda)$. ∎

3. Method for verifying structurally unstable connecting orbits. In this section, we describe an algebraic-topological method for proving the existence of connecting orbits, especially structurally unstable ones. We begin by proving Theorem 1.3.

Proof. Let $S_1 := \mathrm{Inv}(N_1)$, $S_2 := \mathrm{Inv}(N_2)$ and $S := \mathrm{Inv}(N)$. Suppose there exists no connecting orbit from S_1 to S_2.

Choose an arbitrary $x \in S$. Then there is a orbit $\sigma : \mathbb{Z} \to S$ such that $\sigma(0) = x$. Assume $x \in N_2$. Then its forward orbit is contained in N_2 since $f(N_2) \cap N_1 = \emptyset$. If its backward orbit intersects N_1, then the α-limit set of σ is contained in N_1 because $f(N_2) \cap N_1 = \emptyset$ and thus, it follows that σ must be a connecting orbit from S_1 to S_2, contradicting our assumption. Hence $\sigma(\mathbb{Z})$ is contained in N_2 and therefore, $x \in \mathrm{Inv}(N_2)$. Similarly, we have $x \in \mathrm{Inv}(N_1)$ if $x \in N_1$.

This means that S is the disjoint union of invariant subsets S_1 and S_2, and it follows from the additivity of the Conley index (see Theorem 3.22 of [11] or Theorem 1.11 of [13], for example) that $\mathrm{Con}_*(S)$ is the direct sum of $\mathrm{Con}_*(S_1)$ and $\mathrm{Con}_*(S_2)$. This is a contradiction. ∎

Here we note that the Conley index is stable under small perturbations, and so are connecting orbits that can be found by Theorem 1.3. Because, if $\mathrm{Con}_*(S, f) \not\cong \mathrm{Con}_*(S_1, f) \oplus \mathrm{Con}_*(S_2, f)$, then the same relationship holds for every g sufficiently close to f and the corresponding continuations of S_1, S_2 and S. It follows, therefore, that there also exists a connecting orbit between S_1 and S_2 with respect to g.

This means that we can not directly apply Theorem 1.3 to find structurally unstable connecting orbits and in particular homoclinic or heteroclinic tangencies. With this in mind, we make the following simple observation: *Having an unstable connection of codimension one is a stable property under small perturbation of one-parameter families.* Thus, our goal is to apply Theorem 1.3 to a set of maps, instead of an individual map.

Consider a continuous family of maps $f_\lambda : X \to X$ where λ is a real parameter in a closed interval $\Lambda \subset \mathbb{R}$. Assume that there exist families of isolated invariant sets $S_1(\lambda)$, $S_2(\lambda)$ and $S(\lambda)$ continuing over Λ such that $S_1(\lambda)$ and $S_1(\lambda)$ are invariant subsets of $S(\lambda)$ for each λ.

As in the Introduction, we define a map

$$F : (x, \lambda) \mapsto (f_\lambda(x), \lambda) : X \times \Lambda \to X \times \Lambda.$$

Assume that we have isolating neighborhoods N_1, N_2 and N for $S_1 := \bigcup_{\lambda\in\Lambda} S_1(\lambda)$, $S_2 := \bigcup_{\lambda\in\Lambda} S_2(\lambda)$ and $S := \bigcup_{\lambda\in\Lambda} S(\lambda)$, respectively, such that N is the disjoint union of N_1 and N_2.

Now we expect that the map F has a connecting orbit from S_1 to S_2 that is stable under small perturbation of the family F, and hence Theorem 1.3 can be applied. But as shown in the next example, it is often the case that the existence of connecting orbits from S_1 to S_2 is still beyond the scope of Theorem 1.3.

Example 3.1. Consider a one-parameter family of diffeomorphisms f_λ on $\mathbb{R}^3$ illustrated in Figure 3.1. Let $S_1(\lambda) = x$ and $S_2(\lambda) = y$ be the hyperbolic fixed point

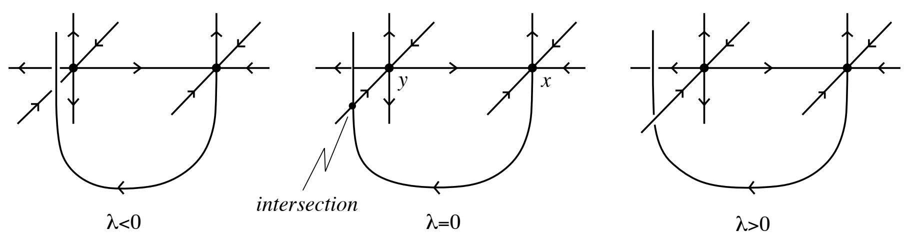

FIG. 3.1. *A hetero-dimensional cycle*

of unstable dimensions 1 and 2, respectively. Assume that $W^u_{f_0}(x)$ intersects $W^s_{f_0}(y)$ at $\lambda = 0$, and therefore there is a connecting orbit from x to y. Let

$$S(\lambda) = \{x\} \cup \{y\} \cup (W^u_{f_\lambda}(x) \cap W^s_{f_\lambda}(y))$$

and N be a sufficiently small compact neighborhood of $S(0)$. Then N is an isolating neighborhood of $S(\lambda)$ for λ close to 0. It is clear that

$$\mathrm{Con}_*(S_i(\lambda)) \cong \mathrm{Con}_*(S_i) \cong \begin{cases} (\mathbb{Z}, 1) & \text{if } * = i \\ (0, 0) & \text{if } * \neq i. \end{cases}$$

An index pair for S is obtained by collapsing the exit set of N. Each component of N is collapsed to a space homotopic to S^1, except for the one that contain x, which is collapsed to a space homotopic to the bouquet of two S^1, and the one that contain y, which is collapsed to a space homotopic to S^2. By computing the shift equivalence class, we have

$$\mathrm{Con}_*(S(\lambda)) \cong \mathrm{Con}_*(S) \cong \begin{cases} (\mathbb{Z}, 1) & \text{if } * = 1, 2 \\ (0, 0) & \text{other.} \end{cases}$$

Observe that although the connecting orbit from $S_1(0)$ to $S_2(0)$ is structurally unstable, having such a connecting orbit is a stable property with respect to a small perturbation of the family. However, this is an example where the converse of Theorem 1.3 does not hold and thus we cannot detect the connecting orbit with this theorem. The problem is that the unstable dimensions of S_1 and S_2 are different, and hence, they have non-trivial Conley index only at different degrees.

Remark that this example illustrates a typical situation that occurs when we consider the projectivization of a homoclinic tangency. Precisely, let p be a hyperbolic saddle fixed point of a surface diffeomorphism f_λ and $x = E^u_p$ and $y = E^s_p$. Then x and y are hyperbolic fixed point of Pf with 1 and 2 dimensional unstable manifolds. There always exists a connection from $y = E^s_p$ to $x = E^u_p$ that is induced from the action of P_pf on P_pX, and there exists a connection from x to y if and only if there exists a homoclinic tangency with respect to p.

To overcome this difficulty, we put an artificial perturbation on F that suspends $\mathrm{Con}_*(S_1)$. Let Λ' be a closed subinterval of Λ such that $\Lambda \setminus \Lambda'$ has two components and suppose $F(N_1) \cap N_2$ is included in $X \times \Lambda'$. This implies that there is no connecting orbit for $\lambda \in \Lambda \setminus \Lambda'$.

Define

$$F'(x, \lambda) = \begin{cases} (f_\lambda(x), \lambda + g(\lambda)) & x \in N_1 \\ (f_\lambda(x), \lambda - g(\lambda)) & x \in N_2 \end{cases}$$

where $g : \Lambda \to \mathbb{R}$ is a continuous function that is negative on the left component of $\Lambda \setminus \Lambda'$, vanishing on Λ' and positive on the right component of $\Lambda \setminus \Lambda'$.

After this perturbation, N_1, N_2 and N remain isolating neighborhoods. Define S'_1 ,S'_2 and S' to be the maximal invariant sets of N_1, N_2 and N with respect to F', respectively. Then by the suspension isomorphism theorem and the homotopy continuation property of the Conley index, we have

$$\mathrm{Con}_*(S'_1, F') = \mathrm{Con}_{*-1}(S_1, F), \quad \mathrm{Con}_*(S'_2, F') = \mathrm{Con}_*(S_2, F).$$

Note that if we apply this construction to Example 3.1, S'_1 has the non-trivial Conley index at degree 2, the same degree at which S'_2 has the non-trivial Conley index.

THEOREM 3.2. *In the above setting, if*

$$\mathrm{Con}_*(S', F') \not\cong \mathrm{Con}_*(S'_1, F') \oplus \mathrm{Con}_*(S'_2, F')$$

then there exists $\lambda_0 \in \Lambda'$ such that there is a connecting orbit from $S_1(\lambda_0)$ to $S_2(\lambda_0)$ under f_{λ_0}.

Proof. By Theorem 1.3, there exists a connection from S'_1 to S'_2 under F'. By our assumption, this connecting orbit must be in $X \times \Lambda'$. But F' and F are identical on Λ', hence the theorem follows. ∎

4. Tangencies in the Hénon family. In this section, we verify the existence of generic homoclinic tangencies in the Hénon family (1.1) by applying the ideas developed in Sections 2 and 3. We explain the steps of the computation in the case of Theorem 1.1, a tangency close to the classical parameter values $a = 1.4$ and $b = 0.3$. With b fixed to 0.3, $H_{a,0.3}$ is now considered to be a one-parameter family with parameter a. For simplicity and to maintain the notation introduced in the earlier sections we write $f_a := H_{a,0.3}$.

We focus on the fixed point

$$p(a) = \left(\frac{-0.7 + \sqrt{0.49 + 4a}}{2}, \frac{-0.7 + \sqrt{0.49 + 4a}}{2} \right)$$

which lies in the first quadrant. By Theorem 2.1, it is sufficient to show the existence of a connecting orbit from $E^u_{p(a)}$ to $E^s_{p(a)}$ for some a. We conclude that the tangency is generic by checking the transversality of $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ using Theorem 2.3.

First we construct isolating neighborhoods N_1, N_2 and N in $PM \times \Lambda = \mathbb{R}^2 \times S^1 \times \mathbb{R}$, with respect to the dynamical system $PF : (x, a) \mapsto (Pf_a(x), a)$. This is done using cubes, i.e. products of closed intervals, in this case, 4-dimensional cubes since TX is homeomorphic to $\mathbb{R}^4$. These isolating neighborhoods are designed so that $S_1 = \mathrm{Inv}(N_1)$ contains $\mathcal{E}^u_\Lambda = \bigcup E^u_{p(a)}$, $S_2 := \mathrm{Inv}(N_2)$ contains $\mathcal{E}^s_\Lambda = \bigcup E^s_{p(a)}$, and $N = N_1 \cup N_2$ contains S_1, S_2 and the connecting orbit of our interest. For simplicity, we write a slice $S \cap (PX \times \{a\})$ of $S \subset PX \times \Lambda$ as $S(a)$, and so forth.

Next we apply the perturbation described in Section 2 to the map PF so that the Conley index of S_1 will be suspended. After perturbation, we have three isolated invariant sets S'_1, S'_2 and S' with respect to PF'.

Here we compute the Conley indexes of S'_1, S'_2 and S' and apply Theorem 3.2. This proves the existence of a connecting orbit from $S_1(a)$ to $S_2(a)$ for some $a \in \Lambda$. Then we show that $S_1(a) = E^u_{p(a)}$ and $S_2(a) = E^s_{p(a)}$. It follows that the connecting orbit we found is from $E^u_{p(a)}$ to $E^s_{p(a)}$, which imply the existence of a tangency with respect to f_a.

Finally, we check that $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ are transversal, and conclude the tangency we found is generic.

The argument above is arranged into the following steps:

Step 1. Construct an initial guess for the location of the connecting orbit.
Step 2. Refine the initial guess up to the desired precision.
Step 3. Modify the refined set to get isolating neighborhoods N_1, N_2 and N.
Step 4. Compute the Conley index and apply Theorem 3.2.
Step 5. Check that $S_1(a) = E^u_{p(a)}$ and $S_2(a) = E^s_{p(a)}$.
Step 6. Check that $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ are transversal.

Before getting into the details of each step, we remark that it is numerically expensive to apply the interval arithmetic to trigonometric and inverse trigonometric functions. Therefore, in the following computations, we choose a piecewise linear coordinate $\theta \in (-\pi, \pi]$ for $P_x M = \mathbb{R}P^1 \cong S^1$. This coordinate is not differentiable, but note that the Conley index theory is still available. To deal with $P(PX \times \Lambda)$, we also take the similar piecewise linear coordinate for $\mathbb{R}P^3$ in the last step.

Step 1. Basically, any method can be used for this step.

In our example, we make use of the software package GAIO in this and next steps. Programs in GAIO are developed for global analysis of invariant objects in dynamical systems by M. Dellnitz, O. Junge and their collaborators. See [2] and the project web page [3]. To construct an initial guess, we simply look at Figure 1.1 and choose cubes that seem to contain the connecting orbit from $E^u_{p(a)}$ to $E^s_{p(a)}$ (Figure 4.1).

Step 2. Next, we refine the initial guess by applying “the subdivision algorithm” [2] of GAIO. In an application of the subdivision algorithm, each cube is divided into two cubes. And then we make a graph map from the multi-valued map induced from PF using the interval arithmetic and remove the cubes which does not contain a connecting orbit or a fixed point of the graph map. Since our computation is

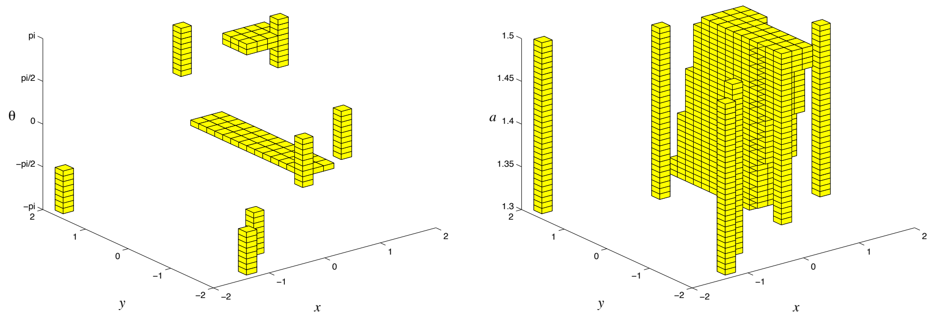

FIG. 4.1. *Our initial guess for the connecting orbit. Left: the projection to the x-y-θ space; Right: the projection to the x-y-a space.*

rigorous, cubes containing a fixed point or a connecting orbit of PF definitely survive this reduction.

After 8 applications of the subdivision and reduction procedure, we get the cubes illustrated in Figure 4.2.

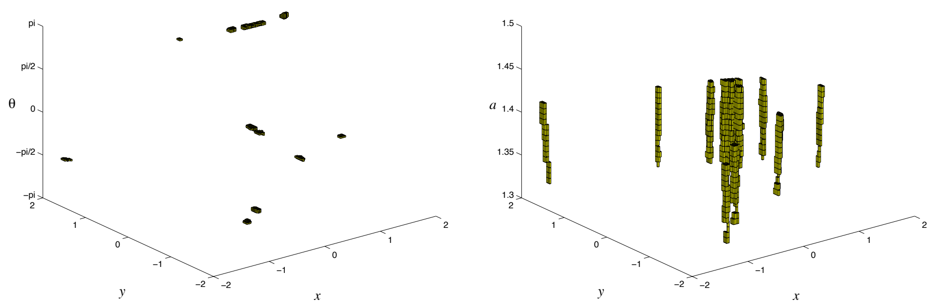

FIG. 4.2. *After* 8 *steps of subdivision and reduction procedure. Left: the projection to the x-y-θ space; Right: the projection to the x-y-a space.*

Cubes after further 8 applications of the procedure are illustrated in Figure 4.3.

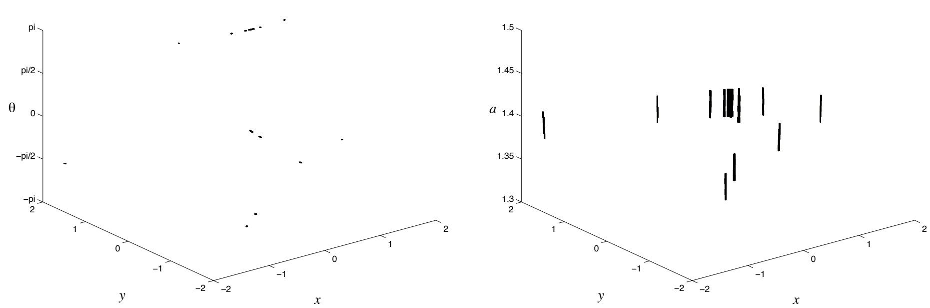

FIG. 4.3. *After* 16 *steps of subdivision and reduction procedure. Left: the projection to the x-y-θ space; Right: the projection to the x-y-a space.*

Note that the range of the parameter value a is getting smaller and smaller during this computation. In our example, we apply this procedure 140 times. The resulting set consists of 9029 cubes and its range of a is smaller than 10^{-10}.

Step 3. Roughly speaking, the algorithm adds cubes to the given set of cubes until it becomes an isolating neighborhood. This is a modification of the algorithm proposed by Junge [7, 8].

Step 4. To construct an index pairs from the isolating neighborhoods found in **Step 3**, we use the combinatorial index pair algorithm (Algorithm 10.86 of [8]). This gives index pairs for S'_1, S'_2 and S'.

Then we apply the Computational Homology Program (`CHomP`, [14]) to compute the Conley index. Application of the program shows that

$$\mathrm{Con}_*(S'_1) = \mathrm{Con}_*(S'_2) = \begin{cases} (0,0) & \text{if } * \neq 2 \\ (\mathbb{Z},1) & \text{if } * = 2 \end{cases}$$

and

$$\mathrm{Con}_*(S') = \begin{cases} (0,0) & \text{if } * \neq 2 \\ (\mathbb{Z}^{59},P) & \text{if } * = 2 \end{cases}$$

where P is a 59 times 59 integer matrix. It can be shown that

$$\mathrm{Con}_2(S') \cong \left(\mathbb{Z}^2, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \not\cong \left(\mathbb{Z}^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \cong \mathrm{Con}_2(S'_1) \oplus \mathrm{Con}_2(S'_2)$$

and therefore, by Theorem 1.3 we conclude that there exists a connecting orbit from $S_1(a)$ to $S_2(a)$ for some $a \in \Lambda'$. In this case, $\Lambda' = [1.392419807915, 1.392419807931]$.

Step 5. We have shown that there exists a parameter value a such that there exists a connecting orbit from $S_1(a)$ to $S_2(a)$. Although $E^u_p(a) \subset S_1(a)$ and $E^s_p(a) \subset S_2(a)$ follows from our construction, it is unknown that whether these set are equal or not. To show these equality, we make use of the Hartman-Grobman linearization theorem.

PROPOSITION 4.1. *Let the origin $0 \in \mathbb{R}^n$ be a hyperbolic fixed point of a diffeomorphism f on $\mathbb{R}^n$ and B a ball of radius r and centered at 0. Choose $0 < \mu < 1$ and $\varepsilon > 0$ so that for each eigenvalue λ of $Tf(0)$ we have $|\lambda| < \mu$ or $|\lambda^{-1}| < \mu$, and $\varepsilon + \mu < 1$ and $\varepsilon < m(Tf(0))$ hold. Here m denotes the minimum norm. If the Lipschitz constant of $f - Tf(0)$ restricted to B is less than $\varepsilon/2$, then* $\mathrm{Inv}(B, f) = \{0\}$.

Proof. Let $g := f - Tf(0)$. Define g' by

$$g'(x) = \begin{cases} g(x) & \text{if } x \in B \\ g(r \cdot x/\|x\|) & \text{if } x \notin B. \end{cases}$$

Then the Lipschitz constant of $g' : \mathbb{R}^n \to \mathbb{R}^n$ is less than ε. Apply the Hartman-Grobman theorem, Theorem 5.7.1 of [15] (Note that Theorem 5.7.1 of [15] gives the estimate on the size of ε). ∎

Since we do not know the exact value of a at which the tangency occurs, we need to show that $S_1(a) = E^u_{p(a)}$ and $S_2(a) = E^s_{p(a)}$ for all $a \in \Lambda'$. Note that since we are using the interval arithmetic, it suffice to check these equalities for a finite number of intervals that cover Λ'.

Using this proposition, we show that $\pi'(S_1(a))$ and $\pi'(S_2(a))$ coincide with the fixed point $p(a)$. Then it is easy to check $S_1(a) = E^u_{p(a)}$ and $S_2(a) = E^s_{p(a)}$ because the dynamics on the fiber $P_{p(a)}\mathbb{R}^2$ is induced from the linear map $T_{p(a)}f_a$. In practice, we first compute ε of the proposition using interval arithmetic. Then check if the condition of the proposition is satisfied with a ball B containing $\pi'(S_1(a))$ or $\pi'(S_2(a))$. In our example of the Hénon map, we have $(f_a - Tf_a(0))(u, v) = (-u^2, 0)$ by the coordinate change $(x, y) = (u + p(a), v + p(a))$. We then can easily check the condition of the proposition. In general, this check may fail. In that case we apply the subdivision algorithm to $S_1(a)$ and $S_2(a)$ to make these sets smaller, and again check if the condition of the proposition holds.

Step 6. Recall that $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ are 2-dimensional manifolds and we need to check that these manifolds are transversal in $\mathbb{R}^2 \times S^1 \times \mathbb{R}$.

For this purpose, we apply the procedure of taking projective bundle once again. That is, we construct $PPF : P(PX\times\Lambda) \to P(PX\times\Lambda)$ from $PF : PX\times\Lambda \to PX\times\Lambda$. Recall that $\mathcal{E}^u_\Lambda$ and $\mathcal{E}^u_\Lambda$ are normally hyperbolic invariant manifolds with respect to PF. Let

$$\mathcal{E}^u\mathcal{E}^u_\Lambda := \{v \in P(PX \times \Lambda) \mid v \in P_x(PX \times \Lambda) \text{ where } x \in \mathcal{E}^u_\Lambda, v \in E^u_x\},$$
$$\mathcal{E}^s\mathcal{E}^s_\Lambda := \{v \in P(PX \times \Lambda) \mid v \in P_x(PX \times \Lambda) \text{ where } x \in \mathcal{E}^s_\Lambda, v \in E^s_x\}.$$

These are the set of all unstable and stable vectors based on a point in $\mathcal{E}^u_\Lambda$ and $\mathcal{E}^s_\Lambda$, respectively. Then it follows that if $W^u_{PF}(\mathcal{E}^u_\Lambda)$ and $W^s_{PF}(\mathcal{E}^s_\Lambda)$ are not transversal and hence share a common subspace at the intersection, there must be a connecting orbit from $\mathcal{E}^u\mathcal{E}^u_\Lambda$ to $\mathcal{E}^s\mathcal{E}^s_\Lambda$ with respect to PPF.

Therefore, it suffice to show the non-existence of such a connecting orbit. This also can be done with interval arithmetics. We subdivide $P(PX \times \Lambda)$ into small cubes and make rigorous coverings (neighborhoods) of $\mathcal{E}^u\mathcal{E}^u_\Lambda$ and $\mathcal{E}^s\mathcal{E}^s_\Lambda$ that consists of cubes. In the case of Hénon map, we can exactly compute $\mathcal{E}^u\mathcal{E}^u_\Lambda$ and $\mathcal{E}^s\mathcal{E}^s_\Lambda$ by hands. In general, we need the help of rigorous interval arithmetics to compute them. Denote these sets of cubes by $\mathcal{U}$ and $\mathcal{S}$, respectively.

By using interval arithmetics, we apply PPF to $\mathcal{U}$ and add the image cubes to $\mathcal{U}$. Then we apply PPF to $\mathcal{U}$ again and repeat this procedure while the number of cubes in $\mathcal{U}$ is increasing. Since the number of the cubes in total space is finite, this procedure stops at some point.

Assume the number of cubes in $\mathcal{U}$ is the same after one application of PPF. Then it follows that the union of cubes in $\mathcal{U}$ is a rigorous covering of the unstable set of $\mathcal{E}^u\mathcal{E}^u_\Lambda$. Then we check if $\mathcal{U} \cap \mathcal{S} = \emptyset$. If this holds, then there can not be a connecting orbit from $\mathcal{E}^u\mathcal{E}^u_\Lambda$ to $\mathcal{E}^s\mathcal{E}^s_\Lambda$ and this is what we wanted to show. If this is not the case, we refine the decomposition of $P(PX \times \Lambda)$ by subdividing all cubes in it and repeat the whole procedure again.

Remark that all the discussion in this section is valid for any b sufficiently close to 0.3. This completes the algorithm to prove Theorem 1.1.

The algorithm for Theorem 1.2 is the same, but the computational cost is different as follows.

	$a = 1.4, b = 0.3$	$a = 1.3, b = -0.3$
Step 2	22.2 min	1.9 min
Step 3	153.9 min	22.5 min
Step 4	26.0 min	50.8 min
Step 6	60.8 min	24.1 min

All the computations are done on a PowerMac G5 (2GHz). Since the orbit of tangency is simpler and hence the number of cubes in the isolating neighborhoods is smaller, the computation for the case $a = 1.3, b = -0.3$ is faster. The only exception is Step 4, the computation of homology. The reason for this is the strong expansion rate of the map, which makes the number of the cubes in the image of the isolating neighborhoods significantly large.

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