On Hyperbolic Plateaus of the Hénon Map

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Abstract

We propose a rigorous computational method to prove the uniform hyperbolicity of discrete dynamical systems. Applying the method to the real Hénon family, we prove the existence of many regions of hyperbolic parameters in the parameter plane of the family.

1 Introduction

Consider the problem of determining the set of parameter values for which the real Hénon map

$$H_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (a - x^2 + by, x) \qquad (a, b \in \mathbb{R})$$

is uniformly hyperbolic. If a dynamical system is uniformly hyperbolic, generally speaking, we can apply the so-called hyperbolic theory of dynamical systems and obtain many results on the behavior of the system. Despite its importance, however, proving hyperbolicity is a difficult problem even for such simple polynomial maps as the Hénon maps.

The first mathematical result about the hyperbolicity of the Hénon map was obtained by [Devaney and Nitecki 79]. They showed that for any fixed b, if a is sufficiently large then the non-wandering set of $H_{a,b}$ is uniformly hyperbolic and conjugate to the full horseshoe map, that is, the shift map of the space of bi-infinite sequences of two symbols.

Later, Davis, MacKay and Sannami [Davis et al. 91] conjectured that besides the uniformly hyperbolic full horseshoe region, there exist some parameter regions in which the non-wandering set of the Hénon map is uniformly hyperbolic and conjugate to a subshift of finite type. For some parameter intervals of the area preserving Hénon family $H_{a,-1}$, they identified the Markov partition by describing the configuration of stable and unstable manifolds (see also [Sterling et al. 99, Hagiwara and Shudo 04]). Although the mechanism of hyperbolicity at these parameter values is clear by their observations, no mathematical proof of the uniform hyperbolicity has been obtained so far.

The purpose of this paper is to propose a general method for proving uniform hyperbolicity of discrete dynamical systems. Applying the method to the Hénon map, we obtain a computer assisted proof of the hyperbolicity of Hénon map on many parameter regions including the intervals conjectured by Davis *et al.*

Our results on the real Hénon map are summarized in the following theorems. We denote by $\mathcal{R}(H_{a,b})$ the chain recurrent set of $H_{a,b}$.

Theorem 1.1. *There exists a set $P \subset \mathbb{R}^2$, which is the union of* 8943 *closed rectangles, such that if $(a, b) \in P$ then $\mathcal{R}(H_{a,b})$ is uniformly hyperbolic. The set P is illustrated in Figure 1 (shaded regions), and the complete list of the rectangles in P is given as a supplemental material to the paper.*

The hyperbolicity of the chain recurrent set implies the $\mathcal{R}$-stability. Therefore, on each connected component of P, no bifurcation occurs in $\mathcal{R}(H_{a,b})$ and hence numerical invariants such as the topological entropy, the number of periodic points, etc., are constant on it. For this reason, we call it a "plateau".

Note that Theorem 1.1 does not claim that a parameter value not in P is a non-hyperbolic parameter. It only guarantees that P is a subset of the uniformly hyperbolic parameter values. We can refine Theorem 1.1 by performing more computations, which yields a set P' of uniformly hyperbolic parameters such that $P \subset P'$.

Since the area-preserving Hénon family is of particular importance, we performed another computation restricted to this one-parameter family and obtained the following.

Theorem 1.2. *If a is in one of the following closed intervals,*

$$
\begin{array}{ll}
[4.5383300781250,\ 4.5385742187500], & [4.5388183593750,\ 4.5429687500000],\\
[4.5623779296875,\ 4.5931396484375], & [4.6188964843750,\ 4.6457519531250],\\
[4.6694335937500,\ 4.6881103515625], & [4.7681884765625,\ 4.7993164062500],\\
[4.8530273437500,\ 4.8603515625000], & [4.9665527343750,\ 4.9692382812500],\\
[5.1469726562500,\ 5.1496582031250], & [5.1904296875000,\ 5.5366210937500],\\
[5.5659179687500,\ 5.6077880859375], & [5.6342773437500,\ 5.6768798828125],\\
[5.6821289062500,\ 5.6857910156250], & [5.6859130859375,\ 5.6860351562500],\\
[5.6916503906250,\ 5.6951904296875], & [5.6999511718750,\ \infty),
\end{array}
$$

then $\mathcal{R}(H_{a,-1})$ is uniformly hyperbolic.

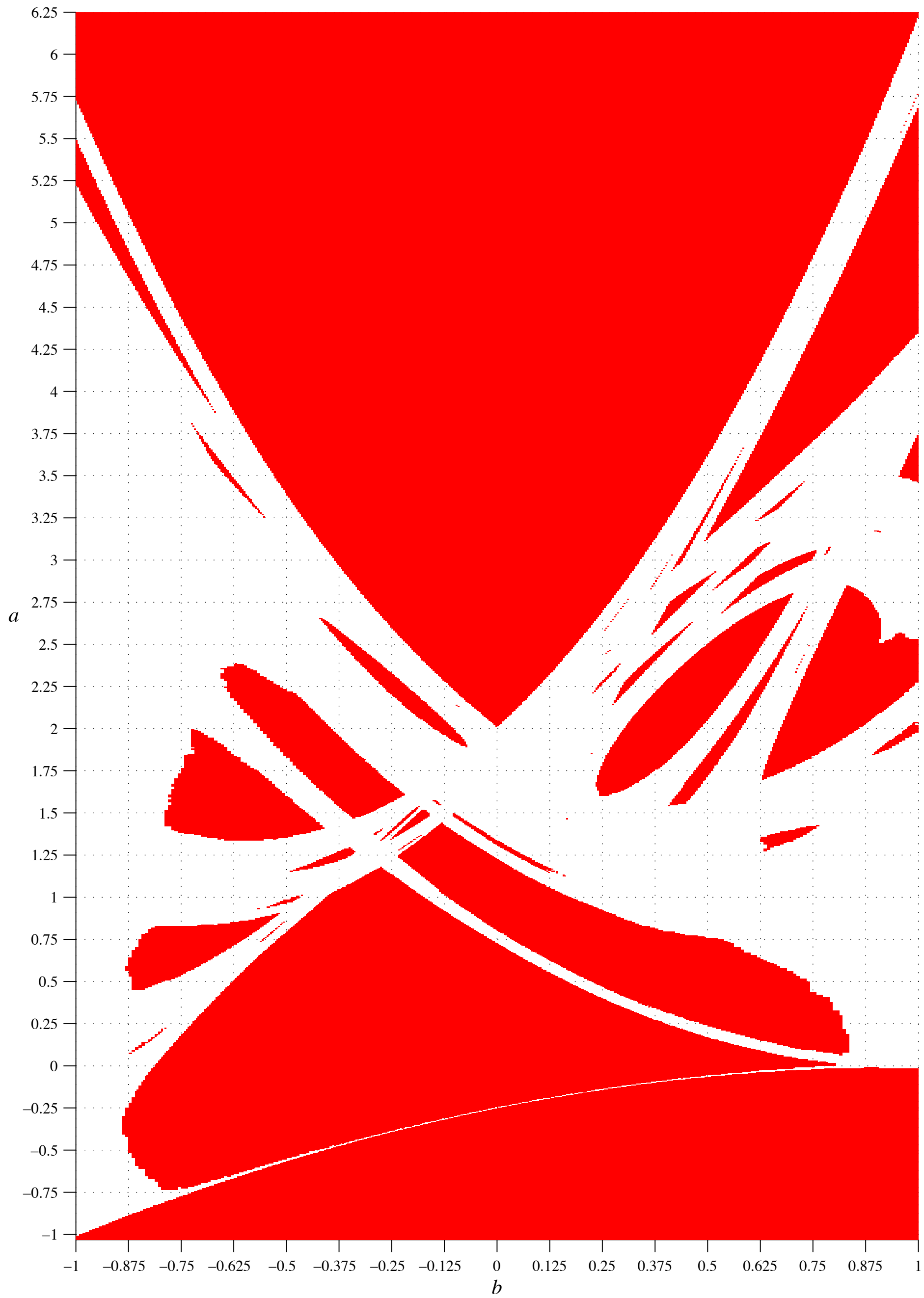

Figure 1: uniformly hyperbolic plateaus

We remark that the three intervals considered to be hyperbolic parameter values by Davis *et al.* appear in Theorem 1.2. Thus we can say that Theorem 1.2 justifies their observations.

It is interesting to compare Figure 1 with the bifurcation diagrams of the Hénon map numerically obtained by [Hamouly and Mira 81], and by [Sannami 89, Sannami 94]. The boundary of P shown in Figure 1 are very close to the bifurcation curves given in these papers.

Recently Cao, Luzzatto and Rios [Cao et al. 05] showed that the Hénon map has a tangency and hence is non-hyperbolic if the parameter is on the boundary of the full horseshoe plateau (see also [Bedford and Smillie 04a, Bedford and Smillie 04b]). This fact and Theorem 1.2 suggests that $H_{a,-1}$ should have a tangency when a is close to 5.699951171875. In fact, we can prove the following theorem using the rigorous computational method developed in [Arai and Mischaikow 05].

Proposition 1.3. *There exists* $a \in [5.6993102, 5.6993113]$ *such that* $H_{a,-1}$ *has a homoclinic tangency with respect to the saddle fixed point on the third quadrant.*

Consequently, Theorem 1.2 and Proposition 1.3 yields the following.

Corollary 1.4. *When we decrease* $a \in \mathbb{R}$ *of the area-preserving Hénon family* $H_{a,-1}$*, the first tangency occurs in* $[5.6993102, 5.699951171875)$.

We remark that Hruska [Hruska 05, Hruska 06] also constructed a rigorous numerical method for proving hyperbolicity of complex Hénon maps. The main difference between our method and Hruska's method is that our method does not prove hyperbolicity directly. Instead, we proves quasi-hyperbolicity, which is equivalent to uniform hyperbolicity under the assumption of chain recurrence. This rephrasing enables us to avoid the computationally expensive procedure of constructing a metric adapted to the hyperbolic splitting. Another peculiar feature of our algorithm is that it is based on the subdivision algorithm (see [Dellnitz and Junge 02a]) and hence effective for inductive search of hyperbolic parameters.

Finally, we remark that the method developed in the paper can also be applied to higher dimensional dynamical systems. In fact, by applying the method to the complex Hénon map, we obtains a proof for Conjecture 1.1 of [Bedford and Smillie 05] (See [Arai 06]).

The structure of the rest of the paper is as follows. The notion of quasi-hyperbolicity will be introduced in §2 and then an algorithm for proving quasi-hyperbolicity will be proposed in §3. In the last section, §4, we apply the method to the Hénon family and obtain Theorem 1.1 and 1.2.

2 Hyperbolicity and Quasi-Hyperbolicity

First we recall the definition of hyperbolicity. Let f be a diffeomorphism on a manifold M and Λ a compact invariant set of f. We denote by $T\Lambda$ the restriction of the tangent bundle TM to Λ.

Definition 2.1. We say that f is *uniformly hyperbolic* on Λ, or Λ is a *uniformly hyperbolic invariant set* of f if $T\Lambda$ splits into a direct sum $T\Lambda = E^s \oplus E^u$ of two Tf-invariant subbundles and there are constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Tf^n|_{E^s}\| < c\lambda^n \quad \text{and} \quad \|Tf^{-n}|_{E^u}\| < c\lambda^n$$

hold for all $n \geq 0$. Here $\|\cdot\|$ denotes a metric on M.

We note that this definition involves many ingredients, constants c and λ, a splitting of $T\Lambda$, and a metric on M. If we try to prove hyperbolicity according to this definition, we must control these objects at the same time, and the algorithm would be rather complicated. Although we can omit the constant c by choosing a suitable metric on M, constructing such a metric is also a difficult problem in general. The situation is the same even if we use the standard "cone fields" argument.

To avoid this computational difficulty, we introduce the notion of quasi-hyperbolicity. Recall that the differential of f induces a dynamical system $Tf : TM \to TM$. By restricting it to the invariant set $T\Lambda$, we obtain $Tf : T\Lambda \to T\Lambda$. An orbit of Tf is called a trivial orbit if it is contained in the zero section of the bundle $T\Lambda$.

Definition 2.2. We say that f is *quasi-hyperbolic* on Λ if $Tf : T\Lambda \to T\Lambda$ has no non-trivial bounded orbit.

This definition is much simpler than that of the uniform hyperbolicity and is a purely topological condition for Tf. It is easy to see that hyperbolicity implies quasi-hyperbolicity. The converse is not true in general, although the hyperbolicity of periodic points and the non-existence of a tangency follows from the quasi-hyperbolicity.

However, when $f|_\Lambda$ is chain recurrent, these two notions coincide.

Theorem 2.3 ([Churchill et al. 77, Sacker and Sell 74])**.** *Assume that $f|_\Lambda$ is chain recurrent, that is, $\mathcal{R}(f|_\Lambda) = \Lambda$. Then f is uniformly hyperbolic on Λ if and only if f is quasi-hyperbolic on it.*

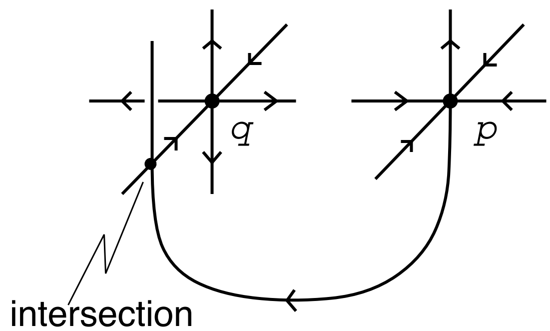

Figure 2: $\Lambda := \{p\} \cup \{q\} \cup (W^u(p) \cap W^s(q))$ is quasi-hyperbolic, but is not uniformly hyperbolic.

Remark 2.4. The assumption of chain recurrence is essential for uniform hyperbolicity. For example, consider two hyperbolic saddle fixed points p and q in $\mathbb{R}^3$, with 1 and 2 dimensional unstable direction respectively. Assume that the unstable manifold $W^u(p)$ of p intersects the stable manifold $W^s(q)$ of q, in a way that the sum of the tangent spaces of these two 1-dimensional manifolds span a 2-dimensional subspace of $\mathbb{R}^3$ (see Figure 2). Let $\Lambda := \{p\} \cup \{q\} \cup (W^u(p) \cap W^s(q))$. Then Λ is quasi-hyperbolic, but clearly not uniformly hyperbolic because it contains fixed points with different unstable dimensions and a connecting orbit between them.

Next, we rephrase the definition of quasi-hyperbolicity in terms of isolating neighborhoods. Recall that a compact set N is an isolating neighborhood (see [Mischaikow and Mrozek 02]) with respect to f if the maximal invariant set

$$\operatorname{Inv}(f, N) := \{x \in N \mid f^n(x) \in N \text{ for all } n \in \mathbb{Z}\}$$

is contained in int N, the interior of N. An invariant set S of f is said to be isolated if there is an isolating neighborhood N such that $\operatorname{Inv}(f, N) = S$.

Note that the linearity of Tf in fibers of $T\Lambda$ implies that if there is a non-trivial bounded orbit of $Tf : T\Lambda \to T\Lambda$, then its multiplication with a constant is also a non-trivial bounded orbit and hence any compact neighborhood N of the zero-section of $T\Lambda$ contains a non-trivial bounded orbit. Therefore, the definition of quasi-hyperbolicity is equivalent to saying that the zero section of the tangent bundle $T\Lambda$ is an isolated invariant set with respect to $Tf : T\Lambda \to T\Lambda$.

Furthermore, it suffice to find an isolating neighborhood that contains the zero section.

Proposition 2.5. *Assume that $N \subset T\Lambda$ is a isolating neighborhood with respect to $Tf : T\Lambda \to T\Lambda$ and N contains the image of the zero-section of $T\Lambda$. Then Λ is quasi-hyperbolic.*

Proof. For a subset S of TM and $\delta \geq 0$, we define $\delta S := \{\delta \cdot v \mid v \in S\}$. By linearity of Tf, if S is Tf-invariant so is δS. Now we assume N is an isolating neighborhood, that is, $\mathrm{Inv}(Tf, N) \subset \mathrm{int}\, N$. A standard compactness argument shows that there is $\delta > 1$ such that $\delta\, \mathrm{Inv}(Tf, N) \subset N$. Since $\delta\, \mathrm{Inv}(Tf, N)$ is Tf-invariant and contained in N, we have $\delta\, \mathrm{Inv}(Tf, N) \subset \mathrm{Inv}(Tf, N)$, by definition of the maximal invariant set. It follows that if $v \in \mathrm{Inv}(Tf, N)$, we have $\delta^n v \in \mathrm{Inv}(Tf, N)$ for all $n \geq 0$. Since $\mathrm{Inv}(Tf, N)$ is compact and hence bounded, v must be the zero vector. This implies that there is no non-trivial bounded orbit of $Tf : T\Lambda \to T\Lambda$. □

3 Algorithm

In this section, we assume that $M = \mathbb{R}^n$ and consider a family of diffeomorphisms $f_a : \mathbb{R}^n \to \mathbb{R}^n$ that depends on r-tuple of real parameters $a = (a_1, \dots, a_r) \in \mathbb{R}^r$. Define $F : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ and $TF : T\mathbb{R}^n \times \mathbb{R}^r \to T\mathbb{R}^n$ by

$$F(x, a) := f_a(x) \quad \text{and} \quad TF(x, v, a) := Tf_a(x, v)$$

where $x \in \mathbb{R}^n$ and $v \in T_x\mathbb{R}^n$.

We denote by $\mathbb{F}$ the set of floating point numbers, or, the set of numbers our computer can handle. Let $\mathbb{IF}$ be the set of intervals whose end-points are in $\mathbb{F}$. Namely,

$$\mathbb{IF} := \{I = [a, b] \subset \mathbb{R} \mid a, b \in \mathbb{F}\}.$$

Similarly, we define a set of n-dimensional cubes by

$$\mathbb{IF}^n := \{I_1 \times \cdots \times I_n \subset \mathbb{R}^n \mid I_i \in \mathbb{IF}\}.$$

Let $X, F \in \mathbb{IF}^n$ and $A \in \mathbb{IF}^r$. We consider these cubes as subspaces of the manifold M, the tangent space of M and the parameter space, respectively. What we want to compute is the image of these cubes under the map F and TF, namely, $F(X \times A)$ and $TF(X \times V \times A)$. Note that these images are not objects of $\mathbb{IF}^n$ nor $\mathbb{IF}^{2n}$ in general. By this fact and the effect of rounding errors, we can not hope that a computer can exactly compute these images. Instead, we require that out computer can enclose these images using elements of $\mathbb{IF}^n$ and $\mathbb{IF}^{2n}$.

Assumption 3.1. There exists a computational method such that for any $X, V \in \mathbb{IF}^n$ and $A \in \mathbb{IF}^r$, it can computes $Y \in \mathbb{IF}^n$ and $W \in \mathbb{IF}^{2n}$ such that

$$F(X \times A) \subset \text{int } Y \quad \text{and} \quad TF(X \times V \times A) \subset \text{int } W$$

hold rigorously.

Obviously, if the outer approximations Y and W in Assumption 3.1 are too large, we can not derive any information of F nor TF. As we will mention in the last section, for many classes of dynamical systems including polynomial maps, the rigorous interval arithmetic can be used to satisfy this assumption and gives effectively good outer approximations.

Let $K \subset \mathbb{R}^n$ be a compact set that contains Λ and $L \subset T\mathbb{R}^n$ the product of K and $[-1,1]^n$. We assume that K is decomposed into a finite union of elements of $\mathbb{IF}^n$, namely,

$$K = \bigcup_{i=1}^{k} K_i \quad \text{where} \quad K_i \in \mathbb{IF}^n.$$

We also decompose the fiber $[-1,1]^n \subset T_x\mathbb{R}^n$ into a finite union of elements of $\mathbb{IF}^n$. By making products of cubes contained in the decompositions of K and $[-1,1]$, we obtain a decomposition of L such as

$$L = \bigcup_{j=1}^{\ell} L_j \quad \text{where} \quad L_j \in \mathbb{IF}^{2n}.$$

By Assumption 3.1, we can compute $Y_i \in \mathbb{IF}^n$ and $W_j \in \mathbb{IF}^{2n}$ such that

$$F(K_i \times A) \subset \text{int } Y_i \quad \text{and} \quad TF(L_j \times A) \subset \text{int } W_j$$

for any $1 \le i \le k$ and $1 \le j \le \ell$.

From this information of Y_i and W_j, we then construct directed graphs $\mathcal{G}(F, K, A)$ and $\mathcal{G}(TF, L, A)$ as follows:

- $\mathcal{G}(F, K, A)$ has k vertices: $\{v_1, v_2, \ldots, v_k\}$.
- There exists an edge from v_p to v_q if and only if $Y_p \cap K_q \neq \emptyset$.

And similarly,

- $\mathcal{G}(TF, L, A)$ has ℓ vertices: $\{w_1, w_2, \ldots, w_\ell\}$.

- There exists an edge from w_p to w_q if and only if $W_p \cap N_q \neq \emptyset$.

The most important property of $\mathcal{G}(F, K, A)$ is that if there exists $x \in K_p$ that is mapped into K_q by f_a for some $a \in A$, then there must be an edge of $\mathcal{G}(F, K, A)$ from v_p to v_q. This property also holds for $\mathcal{G}(TF, L, A)$.

We use these directed graphs to enclose the chain recurrent set of f_a and the maximal invariant set of N. For this purpose, we define the following notions.

Definition 3.2. Let G be a directed graph. The vertices of $\operatorname{Inv} G$, the *invariant set* of G is defined by

$$\{v \in G \mid \exists \text{ bi-infinitely long path through } v\}.$$

The vertices of $\operatorname{Scc} G$, the set of *strongly connected components* of G is

$$\{v \in G \mid \exists \text{ path from } v \text{ to itself}\}.$$

The edges of these graphs are defined to be the restriction of that of G.

Note that by definition, $\operatorname{Scc} G$ is a subgraph of $\operatorname{Inv} G$.

For subgraphs G of $\mathcal{G}(F, K, A)$ and G' of $\mathcal{G}(TF, L, A)$, we define their *geometric representations* $|G| \subset \mathbb{R}^n$ and $|G'| \subset \mathbb{R}^{2n}$by

$$|G| := \bigcup_{v_i \in G} K_i, \quad \text{or} \quad |G'| := \bigcup_{w_j \in G'} L_j,$$

respectively. Obviously, $|\mathcal{G}(F, K, A)| = K$ and $|\mathcal{G}(TF, L, A)| = L$.

Proposition 3.3. *For any $a \in A$,*

$$\operatorname{Inv}(f_a, K) \subset |\operatorname{Inv} \mathcal{G}(F, K, A)| \quad \textit{and} \quad \operatorname{Inv}(Tf_a, L) \subset |\operatorname{Inv} \mathcal{G}(TF, L, A)|.$$

Furthermore, if $\mathcal{R}(f_a) \subset \operatorname{int} K$ holds for all $a \in A$, then we have

$$\mathcal{R}(f_a) \subset |\operatorname{Scc} \mathcal{G}(F, K, A)|$$

for all $a \in A$.

Proof. The claims for maximal invariant sets follow from the construction of $\mathcal{G}(F, K, A)$ and $\mathcal{G}(TF, L, A)$. We only prove $\mathcal{R}(f_a) \subset |\operatorname{Scc} \mathcal{G}(F, K, A)|$. Since $F(K_i \times \{a\}) \subset \operatorname{int} Y_i$ holds for all i and the number of cubes in K is finite, we can choose $\varepsilon > 0$ such that for any i and $x \in K_i$, if y is a point with $d(f_a(x), y) < \varepsilon$ then y must be contained in Y_i. Here d denotes a fixed metric of $\mathbb{R}^n$. This implies that if such y is contained in K_j, there must be an edge from v_i to v_j. Let $x \in \mathcal{R}(f_a)$. From the assumption, there exists p such that $x \in K_p$. Since $\mathcal{R}(f_a) \subset \operatorname{int} K$, we can assume that these is an ε-chain from x to itself that is contained in K, by choosing smaller ε, if necessary. It follows that there must be a path of $\mathcal{G}(F, K, A)$ from v_p to itself and therefore $x \in |\operatorname{Scc}\mathcal{G}(F, K, A)|$. This proves the claim. ☐

For the computation of $\operatorname{Inv} G$, the algorithm of [Szymczak 97] can be used. There is also an algorithm for computing $\operatorname{Scc} G$ that is standard in the algorithmic graph theory (see [Sedgewick 83], for example).

Now we can describe the algorithm to prove the quasi-hyperbolicity. The algorithm involves the subdivision algorithm [Dellnitz and Junge 02a]. Roughly speaking, this means that if we fails to prove quasi-hyperbolicity, then we subdivide all of the cubes in K and L to have a better approximation of the invariant set, and repeat the whole step until we succeed the proof.

In the following, we first develop an algorithm for a fixed set of parameter values. That is, we fix the set A and try to check if $\mathcal{R}(f_a)$ is quasi-hyperbolic for all $a \in A$. Note that we do not exclude the case A contains only one parameter value, namely, $A = \{a\}$ where a is a r-tuple of floating point numbers.

Algorithm 3.4. (for proving quasi-hyperbolicity for all all $a \in A$)

1. Find K such that $\mathcal{R}(f_a) \subset \operatorname{int} K$ holds for all $a \in A$ and let $L := K \times [-1, 1]^n$.
2. Compute $\operatorname{Scc}\mathcal{G}(F, K, A)$ and replace K with $|\operatorname{Scc}\mathcal{G}(F, K, A)|$.
3. Replace L with $L \cap (K \times [-1, 1]^n)$.
4. Compute $\operatorname{Inv}\mathcal{G}(TF, L, A)$.
5. If $|\operatorname{Inv}\mathcal{G}(TF, K, A)| \subset K \times \operatorname{int}[-1, 1]^n$ then stop.
6. Otherwise, replace L with $|\operatorname{Inv}\mathcal{G}(TF, L, A)|$ and refine the decomposition of K and L by bisecting all cubes in them. Then goto step 2.

Theorem 3.5. *If Algorithm* 3.4 *stops, then f_a is quasi-hyperbolic on $\mathcal{R}(f_a)$ for every $a \in A$.*

Proof. Assume that Algorithm 3.4 stops and choose $a \in A$. Let

$$N_a = L \cap (\mathcal{R}(f_a) \times [-1, 1]^n).$$

Then N_a contains the zero-section of $T\mathcal{R}(f_a)$. By Proposition 2.5, it suffice to show that N_a is an isolating neighborhood with respect to $Tf_a : T\mathcal{R}(f_a) \to T\mathcal{R}(f_a)$. Since the algorithm stops,

$$\mathrm{Inv}(Tf_a, N_a) \subset \mathrm{Inv}(Tf_a, L) \subset |\,\mathrm{Inv}\,\mathcal{G}(TF, L)| \subset K \times \mathrm{int}\,[-1,1]^n.$$

Then it follows from $\mathrm{Inv}(Tf_a, N_a) \subset N_a \subset \mathcal{R}(f_a) \times [-1,1]^n$ that

$$\mathrm{Inv}(Tf_a, N_a) \subset \mathcal{R}(f_a) \times \mathrm{int}\,[-1,1]^n.$$

But $\mathcal{R}(f_a) \times \mathrm{int}\,[-1,1]^n$ is the interior of N_a with respect to $T\mathcal{R}(f_a)$ and this proves $\mathrm{Inv}(Tf_a, N_a) \subset \mathrm{int}\; N_a$. □

In other words, if A contains a non-quasi-hyperbolic parameter value, then Algorithm 3.4 never stops. Therefore, if we want to apply the method for a large family of diffeomorphism, the algorithm should involve an automatic selection of parameter values.

We can also use the subdivision algorithm to realize such a procedure. That is, we will inductively decompose A into a finite union of elements of $\mathbb{IF}^r$ and remove cubes in which the hyperbolicity is proved. We denote by $\mathcal{A}$ the set of cubes in the decomposition of A.

Algorithm 3.6. (adaptive selection of quasi-hyperbolic parameters)

1. Find K such that $\mathcal{R}(f_a) \subset K$ holds for all $a \in A$.
2. Let $\mathcal{A} = \{A_0\}$ where $A_0 = A$ and $K_0 = K$, $L_0 = K_0 \times [-1,1]^n$.
3. Choose a cube $A_i \in \mathcal{A}$ according to the "selection rule".
4. Apply step 2, 3, and 4 of Algorithm 3.4 with $A = A_i$, $K = K_i$ and $L = L_i$.
5. If $|\,\mathrm{Inv}\,\mathcal{G}(TF, L_i)| \subset \mathrm{int}\; L_i$ then remove A_i from $\mathcal{A}$ and goto step 2.
6. Otherwise, bisect A_i into two cubes A_j, A_k. Remove A_i from $\mathcal{A}$ and add A_j, A_k to $\mathcal{A}$. Put $K_j = K_k = K_i$ and $L_j = L_k = \mathrm{Inv}\,\mathcal{G}(TF, L_i)$ then goto step 2.

This algorithm does not stop neither if there is a non quasi-hyperbolic parameter in A. But it follows from Theorem 3.5 that if the cube A_i is removed in the procedure of Algorithm 3.6, then A_i consists of quasi-hyperbolic parameter values.

We did not specify the "selection rule" that appears in step 2 of Algorithm 3.6. Various rules can be applied and the effectiveness of a rule depends on the case.

One example of such a rule is selecting A_i such that N_i and K_i consist of smaller number of cubes. Since the computational cost of the algorithm depends on the number of cubes, this rule implies that our computation will be concentrated on parameter values on which the computation is relatively fast. By applying this rule, we can avoid wasting too much time trying to prove the hyperbolicity for apparently non-hyperbolic parameter values. This rule works sufficiently well for general purposes.

The problem with this rule is that sometimes computation is focused only on parameters with small invariant set, for example, where $\mathcal{R}(f_a)$ is a single fixed point. If this is the case, the most of the computation will be done on parameter cubes close to the bifurcation curve of the fixed point. To avoid this, we can use the number of cubes multiplied by the subdivision depth of A_i instead of the number of cubes itself.

Or, we can distribute our computational effort across the whole of the parameter space equally by simply selecting all cubes in $\mathcal{A}$ sequentially.

4 Application to the Hénon Map

In this section, we apply the method developed in §2 and §3 to the chain recurrent set $\mathcal{R}(H_{a,b})$ of the Hénon family.

In order to apply the algorithm, we must know a priori the size of $\mathcal{R}(H_{a,b})$. Further, to apply Theorem 2.3, we need to check that the dynamics restricted to $\mathcal{R}(H_{a,b})$ is chain recurrent.

First we recall the numbers defined in [Devaney and Nitecki 79]. Let

$$R(a,b) := \frac{1}{2}(1 + |b| + \sqrt{(1+|b|)^2 + 4a}),$$
$$S(a,b) := \{(x,y) \in \mathbb{R}^2 : |x| \le R(a,b), |y| \le R(a,b)\}.$$

Then we can prove the following.

Lemma 4.1. *The chain recurrent set $\mathcal{R}(H_{a,b})$ is contained in $S(a,b)$. And $H_{a,b}$ restricted to $\mathcal{R}(H_{a,b})$ is chain recurrent.*

Proof. If $x \notin S(a,b)$, we can choose $\varepsilon_0 > 0$ so small that if $\varepsilon < \varepsilon_0$ then all ε-chains thorough x must diverge to infinity and hence, x can not be chain recurrent (this is a special case of Corollary 2.7 of [Bedford and Smillie 91]).

The proof for the second claim is the same as that for the compact case (see [Robinson 99] for example), because we can choose a compact neighborhood S' of $S(a, b)$ and $\varepsilon_0 > 0$ such that if $\varepsilon < \varepsilon_0$ then all ε-chain from $x \in \mathcal{R}$ to x must be contained in S'. □

In the case of Hénon map, Assumption 3.1 can be satisfied with rigorous interval arithmetic on a CPU that satisfies IEEE754 standard for binary floating-point arithmetic. This is also the case for an arbitrary polynomial map of $\mathbb{R}^n$.

We remark that we only need to consider the case $b \in [-1, 1]$. Because, the inverse of the Hénon map $H_{a,b}$ is again conjugate to the Hénon map $H_{a/b^2,1/b}$, whose Jacobian is $1/b$, and the hyperbolicity of a diffeomorphism is equivalent to that of the inverse map. Further, we can restrict our computation to the case $(a, b) \in [-1, 12] \times [-1, 1]$, for otherwise it follows from the proof of [Devaney and Nitecki 79] that $\mathcal{R}(H_{a,b})$ is hyperbolic or empty.

Therefore, we start with $A := [-1, 12] \times [-1, 1]$, $K := [-8, 8] \times [-8, 8]$ and $L = K \times [-1, 1]^2$. Then Lemma 4.1 implies $\mathcal{R}(H_{a,b}) \subset \operatorname{int} K$ holds for all $(a, b) \in A$. With this initial data, Theorem 1.1 is proven by applying Algorithm 3.6.

To obtain Theorem 1.2, we fix $b = -1$ and start the computation with $A := [4, 12]$. The sets K and L are the same as the computation for Theorem 1.1.

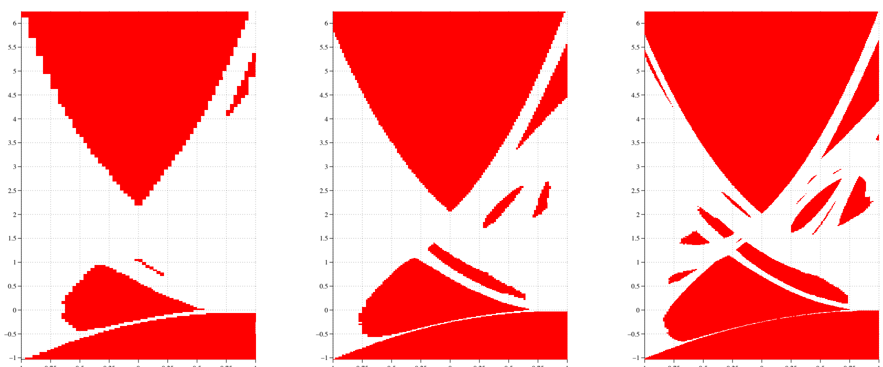

Figure 3: results after 1, 10 and 100 hour computation (from left to right)

Finally, we mention the computational cost of the method. To achieve Theorem 1.1, we need 1000 hours of computation using 2GHz PowerPC 970

CPU. With the same CPU, 260 hours were used for Theorem 1.2. Figure 3 shows the intermediate results obtained after 1, 10 and 100 hours computation toward Theorem 1.2. We remark that as these figures suggest, almost all computations time was spent on the parameter values close to bifurcation curves.

All of the source codes used in these computations are available at the home page of the author (`http://www.math.kyoto-u.ac.jp/~arai/`). The data structure and the subdivision algorithm are implemented in the GAIO package `http://math-www.uni-paderborn.de/~agdellnitz/gaio/` [Dellnitz and Junge 02a] are used in these programs. For the interval arithmetic, we use the package CAPD (`http://capd.wsb-nlu.edu.pl/`). You can also use PROFIL/BIAS interval arithmetic package for this purpose (`http://www.ti3.tu-harburg.de/knueppel/profil`).

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